

Projectively flat general (α, β) -metrics with constant flag curvature

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January 17, 2015

Abstract

In this paper we study the flag curvature of a new class of Finsler metrics called general (α, β) -metrics, which are defined by a Riemannian metric α and a 1-form β . The classification of such metrics with constant flag curvature are completely determined under some suitable conditions, which make them locally projectively flat. As a result, we construct many new projectively flat Finsler metrics with flag curvature 1, 0 and -1 in Section 9, all of which are of singularity at some directions. The simplest one is given by $F = \frac{(b\alpha + \beta)^2}{\alpha}$ where $b = \|\beta\|_\alpha$.

1 Introduction

In Finsler geometry, many important Finsler metrics with constant flag curvature are locally projectively flat. For example, the generalized Funk metrics

$$F = \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y \rangle^2}}{1 - |x|^2} \pm \left\{ \frac{\langle x, y \rangle}{1 - |x|^2} + \frac{\langle a, y \rangle}{1 + \langle a, x \rangle} \right\} \quad (1.1)$$

are locally projectively flat with constant flag curvature $K = -\frac{1}{4}$, where a is a constant vector[9]. (1.1) belong to a special class of Finsler metrics called Randers metrics given in the form $F = \alpha + \beta$, where α is a Riemannian metric and β is a 1-form. Moreover, the generalized Berwald's metrics

$$F = \frac{((1 + \langle a, x \rangle)(\sqrt{(1 - |x|^2)|y|^2 + \langle x, y \rangle^2} + \langle x, y \rangle) + (1 - |x|^2)\langle a, y \rangle)^2}{(1 - |x|^2)^2 \sqrt{(1 - |x|^2)|y|^2 + \langle x, y \rangle^2}} \quad (1.2)$$

are also locally projectively flat with constant flag curvature $K = 0$ [10]. (1.2) belong to the so-called square metrics given in the form $F = \frac{(\alpha + \beta)^2}{\alpha}$. [11]

Both Randers metrics and square metrics belong to the metrical category called (α, β) -metrics, which are given in the form $F = \alpha \phi(\frac{\beta}{\alpha})$, where $\phi(s)$ is a smooth function. In 2007, Li-Shen proved that except Riemannian metrics and locally Minkowskian metrics, any locally projectively flat (α, β) -metric with constant flag curvature K is either locally isometric to a generalized Funk metric after a scaling when $K < 0$, or locally isometric to a generalized Berwald's metric after a scaling when $K = 0$ [5].

Randers metrics can be expressed in another famous form

$$F = \frac{\sqrt{(1 - b^2)\alpha^2 + \beta^2}}{1 - b^2} + \frac{\beta}{1 - b^2}, \quad (1.3)$$

where $b := \|\beta_x\|_\alpha$ is the length of β . Combining with Bao-Robles-Shen's well-known classification result[2] and the related discussions in [9], one can see that a Randers metric is locally projectively flat and of constant flag curvature if and only if α in (1.3) is locally projectively flat and β is closed and homothetic with respect to α . Note that Beltrami's theorem says that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. So the above fact means that α and β satisfy

$${}^\alpha R^i_j = \mu(\alpha^2 \delta^i_j - y^i y_j), \quad b_{i|j} = c a_{ij},$$

⁰ *Keywords:* Finsler metric, general (α, β) -metric, flag curvature, projective flatness, deformation.
Mathematics Subject Classification: 53B40, 53C60.

where c is a constant.

Square metrics can also be expressed in another form

$$F = \frac{(\sqrt{(1-b^2)\alpha^2 + \beta^2} + \beta)^2}{(1-b^2)^2 \sqrt{(1-b^2)\alpha^2 + \beta^2}}. \quad (1.4)$$

The first author showed that a square metric is locally projectively flat if and only if α in (1.4) is locally projectively flat and β is closed and conformal with respect to α [15], i.e.,

$${}^\alpha R^i_j = \mu(\alpha^2 \delta^i_j - y^i y_j), \quad b_{i|j} = c(x) a_{ij}, \quad (1.5)$$

where $c(x)$ is a scalar function on the manifold. Later on, Z. Shen and the first author proved that a square metric is an Einstein metric if and only if α and β satisfy

$${}^\alpha Ric = 0, \quad b_{i|j} = c a_{ij},$$

where c is a constant[11].

One can see from the above facts that the expressions (1.3) and (1.4) have the advantage of clearly illuminating the underlying geometry, although they are more complicated in algebraic form. This is a common phenomenon about (α, β) -metrics[13, 14]. Actually, both Randers metrics and square metrics belong to a larger class of Finsler metrics called general (α, β) -metrics, which are also defined by a Riemannian metric α and a 1-form β and given in the form

$$F = \alpha \phi \left(b^2, \frac{\beta}{\alpha} \right), \quad (1.6)$$

where $\phi(b^2, s)$ is a smooth function[16].

If $\phi = \phi(s)$ is independent of b^2 , then $F = \alpha \phi(\frac{\beta}{\alpha})$ is a (α, β) -metric. If $\alpha = |y|$, $\beta = \langle x, y \rangle$, then $F = |y| \phi(|x|^2, \frac{\langle x, y \rangle}{|y|})$ is the so-called spherically symmetric Finsler metrics. Moreover, general (α, β) -metrics include part of Bryant's metrics and part of fourth root metrics. That is to say, general (α, β) -metrics make up of a much large class of Finsler metrics, which makes it possible to find out more Finsler metrics to be of great properties. For example, in (α, β) -metrics we can't find out any non-Ricci flat Einstein metric unless it is of Randers type[4]. The main reason is that the category of (α, β) -metrics is a little small. If we search Einstein metrics in general (α, β) -metrics, then it is not hard to find out metrics with positive and negative Ricci constant[12].

Come back to our discussions. It is clear that the corresponding functions $\phi(b^2, s)$ of (1.3) and (1.4) are given by $\phi = \frac{\sqrt{1-b^2+s^2}}{1-b^2} + \frac{s}{1-b^2}$ and $\phi = \frac{(\sqrt{1-b^2+s^2}+s)^2}{(1-b^2)^2 \sqrt{1-b^2+s^2}}$ respectively. Moreover, both of them satisfy the following PDE:

$$\phi_{22} = 2(\phi_1 - s\phi_{12}). \quad (1.7)$$

Here ϕ_1 means the derivation of ϕ with respect to the first variable b^2 .

In fact, the first author proved that when dimension $n \geq 3$, every non-trivial locally projectively flat (α, β) metric can be reexpressed as a new form $F = \alpha \phi(b^2, s)$ such that the corresponding function ϕ satisfies (1.7), and at the same time α and β satisfy (1.5)[15]. We believe that it also holds for general (α, β) -metrics, although we don't still know how to prove it by now. Until now, it is known that if (1.5) holds, then the general (α, β) -metric $F = \alpha \phi(b^2, s)$ is locally projectively flat if and only if ϕ satisfies (1.7)[12].

The aim of this paper is to study general (α, β) -metrics with constant flag curvature. It's worth mentioning here that L. Zhou proved an interesting result in 2010: if a square metric is of constant flag curvature, then it must be locally projectively flat[17]. It is not true for Randers metrics, because there are many Randers metrics with constant flag curvature which are not locally projectively flat actually[2]. Even so, we have reason to believe that Zhou's result holds for any non-Randers type general (α, β) -metrics. More specifically, we conjecture that except Randers metrics, there does not exist any non locally projectively flat regular general (α, β) -metric to be of constant flag curvature. Randers metrics are very particular, the key reason is that any Randers metric will still turn to be a Randers metric after navigation transformation[2], but for a non-Randers type general (α, β) -metric, it will not turn to be a general (α, β) -metric after navigation transformation in general[16].

Hence, we will discuss our problem under the assumption that α and β satisfy the conditions (1.5), and ϕ satisfies the condition (1.7). In this case, the corresponding general (α, β) -metric must be locally projectively

flat. Moreover, Lemma 3.1 shows that the conformal factor $c(x)$ in (1.5) must satisfy $c^2 = \kappa - \mu b^2$ for some constant κ .

To be more clear, let's illustrate our assumption in this paper again:

Assumption: α , β and ϕ satisfy

$${}^\alpha R^i_j = \mu(\alpha^2 \delta^i_j - y^i y_j), \quad b_{i|j} = (\kappa - \mu b^2) a_{ij}, \quad \phi_{22} = 2(\phi_1 - s\phi_{12}) \quad (1.8)$$

respectively.

We believe that such conditions are natural. Our main reason is that all the known general (α, β) -metrics with constant flag curvature, including Bryant's metrics which are not discussed above, can be reexpressed to fit it.

The general (α, β) -metrics $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ with constant flag curvature under our assumption can be completely solved. Firstly, we have the following equivalent characterization.

Theorem 1.1. *Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α, β) -metric on an n -dimensional manifold M with $n \geq 3$, where α , β and ϕ satisfy (1.8). Then F is of constant flag curvature K if and only if the function $\phi = \phi(b^2, s)$ satisfies the following PDE:*

$$(\kappa - \mu b^2) [\psi^2 - (\psi_2 + 2s\psi_1)] + \mu s\psi + \mu = K\phi^2, \quad (1.9)$$

where $\psi := \frac{\phi_2 + 2s\phi_1}{2\phi}$.

The Riemannian metrics α and 1-forms β satisfying (1.5) have already been determined completely (see (3.1)). According to Theorem 1.1, in order to determine the general (α, β) -metrics with constant flag curvature under our assumption, we only need to solve Equation (1.7) and (1.9).

The case when $\kappa = 0$ and $\mu = 0$ is trivial, because in the case α is locally Euclidian and β is parallel with respect to α . As a result, $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ is locally Minkowskian and hence flat automatically for any suitable function $\phi(b^2, s)$.

When $\kappa \neq 0$ and $\mu = 0$, we have the following result.

Theorem 1.2. *Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α, β) -metric on an n -dimensional manifold M with $n \geq 3$, where α , β and ϕ satisfy (1.8) with $\kappa \neq 0$ and $\mu = 0$. Then F is of constant flag curvature K if and only if ϕ is given by one of the forms:*

$$\phi = \frac{1}{2\sqrt{-\sigma}} \frac{1}{\sqrt{C - b^2 + s^2 \pm s}}, \quad (1.10)$$

$$\phi = \frac{q(u)}{q^2(u)(Dq(u) + v)^2 + \sigma}, \quad (1.11)$$

where $\sigma := K/\kappa$, $u := b^2 - s^2$ and $v := s$, the function $q(u)$ satisfies the following equation:

$$D^2 q^4 + (u - C)q^2 - \sigma = 0,$$

where C and D are constants.

The case when $\kappa \neq 0$ and $\mu \neq 0$ can be reduced to the above case by some special deformations. See Section 3 and Example 8.5 for details.

The case when $\kappa = 0$ and $\mu \neq 0$ is very special, and we have the following result.

Theorem 1.3. *Let $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α, β) -metric on an n -dimensional manifold M with $n \geq 3$, where α , β and ϕ satisfy (1.8) with $\kappa = 0$ and $\mu \neq 0$. Then F is of constant flag curvature K if and only if ϕ is given by:*

$$\phi(u, v) = \frac{2q(u)(\sqrt{u + v^2} \pm v)^2}{[q(u)(\sqrt{u + v^2} \pm v)^2 + p(u)]^2 + \tau}, \quad (1.12)$$

where $\tau := -K/\mu$, $u := b^2 - s^2$ and $v := s$, the functions $p(u)$ and $q(u)$ are given by one of the forms:

$$p(u) = \pm\sqrt{-\tau}, \quad q(u) = \pm \frac{(C \pm \sqrt{C^2 + 8pu})^2}{4u^2} \quad (1.13)$$

or

$$p(u) = \pm \sqrt{\frac{-(C^2 - D)\tau - C(C\tau - 2u) \pm \sqrt{D(C\tau - 2u)^2 - D(C^2 - D)\tau^2}}{2(C^2 - D)}}, \quad (1.14)$$

$$q(u) = \frac{p^2 + \tau - upp' \pm \sqrt{(p^2 + \tau - upp')^2 - (p^2 + \tau)u^2p'}}{u^2p'}, \quad (1.15)$$

where C and D are constants.

By Theorem 1.3, we can obtain some new Finsler metrics with constant flag curvature 1, 0 and -1 . For example, it is easy to check that $\phi(b^2, s) = (b + s)^2$ satisfies Equations (1.7) and (1.9) with $\kappa = 0$, $\mu \neq 0$ and $K = 0$, so

$$F = \frac{(b\alpha + \beta)^2}{\alpha}$$

is projectively flat and of vanishing flag curvature, where

$$\alpha = \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y \rangle^2}}{1 + \mu|x|^2}, \quad \beta = \frac{\lambda\langle x, y \rangle + (1 + \mu|x|^2)\langle a, y \rangle - \mu\langle a, x \rangle\langle x, y \rangle}{(1 + \mu|x|^2)^{\frac{3}{2}}}.$$

with additionally $\lambda^2 + \mu|a|^2 = 0$, which makes α and β satisfy (1.8) with $\kappa = 0$. One can find more examples in Section 9.

Notice that $\phi - s\phi_2 = b^2 - s^2$ and $\phi - s\phi_2 + (b^2 - s^2)\phi_{22} = 3(b^2 - s^2)$, so such metrics are non-regular at the directions $(y^i) = \pm(b^i)$. Moreover, $\phi = 0$ when $(y^i) = -(b^i)$. Actually, all the metrics determined by Theorem 1.3 have the same singularity. Hence, we have

Corollary 1.4. *When $n \geq 3$, all the non-trivial regular general (α, β) -metrics $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ with constant flag curvature satisfying (1.8) are completely determined by Theorem 1.2.*

2 Preliminaries

Let F be a Finsler metric on an n -dimensional manifold M and G^i be the geodesic coefficients of F , which are defined by

$$G^i = \frac{1}{4}g^{il} \{ [F^2]_{x^k y^l} y^k - [F^2]_{x^l} \},$$

where $(g^{ij}) := (\frac{1}{2}[F^2]_{y^i y^j})^{-1}$. For a Riemannian metric, the spray coefficients are determined by its Christoffel symbols as $G^i(x, y) = \frac{1}{2}\Gamma_{jk}^i(x)y^j y^k$.

For any $x \in M$ and $y \in T_x M \setminus \{0\}$, the Riemann curvature tensor $R_y = R^i_j \frac{\partial}{\partial x^i} \otimes dx^j$ of F is defined by

$$R^i_j = 2 \frac{\partial G^i}{\partial x^j} - \frac{\partial^2 G^i}{\partial x^k \partial y^j} y^k + 2G^k \frac{\partial^2 G^i}{\partial y^k \partial y^j} - \frac{\partial G^i}{\partial y^k} \frac{\partial G^k}{\partial y^j}.$$

The value as follows

$$K(P, y) := \frac{g_y(R_y(u), u)}{g_y(y, y)g_y(u, u) - [g_y(y, u)]^2}$$

is called the flag curvature of the flag plane $P = \text{span}\{y, u\} \subset T_x M$ along the direction y . When F is Riemannian, $K(P, y) = K(P)$ is independent of $y \in P$ and it is just the sectional curvature of P in Riemann geometry. F is said to be of constant flag curvature if for any $y \in T_x M$, the flag curvature $K(P, y) = K$ is a constant, that is equivalent to the following system of equations in a local coordinate system (x^i, y^i) in TM ,

$$R^i_j = KF^2(\delta^i_j - F^{-1}F_{y^j}y^i).$$

On the other hand, a Finsler metric F on a manifold M is said to be locally projectively flat if at any point, there is a local coordinate system (x^i) in which the geodesics are straight lines as point sets. In this case, the

spray coefficients are in the form $G^i = Py^i$, where $P = P(x, y)$ given by $P = \frac{F_{x^k} y^k}{2F}$ is called the projective factor of F . For a projectively flat Finsler metric F , the flag curvature is given by

$$K = \frac{P^2 - P_{x^k} y^k}{F^2}. \quad (2.1)$$

By definition, a general (α, β) -metric is given by (1.6) where $\phi = \phi(b^2, s)$ is a smooth function defined on the domain $|s| \leq b < b_o$ for some positive number (maybe infinity) b_o , α is a Riemannian metric and β is a 1-form with $b < b_o$. When $n \geq 3$, $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ is a regular Finsler metric for any α and β with $b < b_o$ if and only if $\phi(b^2, s)$ satisfies

$$\phi - s\phi_2 > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \quad |s| \leq b < b_o.$$

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Denote the coefficients of the covariant derivative of β with respect to α by $b_{i|j}$, and let

$$\begin{aligned} r_{ij} &= \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r_{00} = r_{ij}y^i y^j, \quad s^i_0 = a^{ij}s_{jk}y^k, \\ r_i &= b^j r_{ji}, \quad s_i = b^j s_{ji}, \quad r_0 = r_i y^i, \quad s_0 = s_i y^i, \quad r^i = a^{ij}r_j, \quad s^i = a^{ij}s_j, \quad r = b^i r_i, \end{aligned}$$

where $(a^{ij}) := (a_{ij})^{-1}$ and $b^i := a^{ij}b_j$. It is easy to see that β is closed if and only if $s_{ij} = 0$.

Lemma 2.1. [16] *The geodesic coefficients G^i of a general (α, β) -metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ are given by*

$$\begin{aligned} G^i &= {}^\alpha G^i + \alpha Q s^i_0 + \left\{ \Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha \Omega(r_0 + s_0) \right\} \frac{y^i}{\alpha} \\ &\quad + \left\{ \Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha \Pi(r_0 + s_0) \right\} b^i - \alpha^2 R(r^i + s^i), \end{aligned} \quad (2.2)$$

where ${}^\alpha G^i$ are the geodesic coefficients of α , and

$$\begin{aligned} Q &= \frac{\phi_2}{\phi - s\phi_2}, \quad R = \frac{\phi_1}{\phi - s\phi_2}, \quad \Omega = \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi} \Pi, \\ \Theta &= \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{2\phi(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \quad \Psi = \frac{\phi_{22}}{2(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}, \quad \Pi = \frac{(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22}}{(\phi - s\phi_2)(\phi - s\phi_2 + (b^2 - s^2)\phi_{22})}. \end{aligned}$$

Note that ϕ_1 means the derivation of ϕ with respect to the first variable b^2 .

Finally, it is known that if the geodesic spray coefficients of a Finsler metric F are given by

$$G^i = {}^\alpha G^i + Q^i,$$

then the Riemann curvature tensor of F are related to that of α and given by

$$R^i_j = {}^\alpha R^i_j + 2Q^i_{|j} - y^m Q^i_{|m.j} + 2Q^m Q^i_{.m.j} - Q^i_{.m} Q^m_{.j}, \quad (2.3)$$

where “ $|$ ” and “ $.$ ” denote the horizontal covariant derivative and vertical covariant derivative with respect to α respectively, i.e., $*.i = \frac{\partial *}{\partial y^i}$.

3 Constant sectional curvature Riemannian metrics and their conformal 1-forms

According to [15], if α and β satisfy (1.5), then there is a local coordinate system in which

$$\alpha = \frac{\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y \rangle^2}}{1 + \mu|x|^2}, \quad \beta = \frac{\lambda\langle x, y \rangle + (1 + \mu|x|^2)\langle a, y \rangle - \mu\langle a, x \rangle\langle x, y \rangle}{(1 + \mu|x|^2)^{\frac{3}{2}}}. \quad (3.1)$$

In this case,

$$b_{i|j} = \frac{\lambda - \mu\langle a, x \rangle}{\sqrt{1 + \mu|x|^2}} a_{ij}. \quad (3.2)$$

One can check directly

$$c^2(x) = \lambda^2 + \mu|a|^2 - \mu b^2. \quad (3.3)$$

Hence, we immediately have

Lemma 3.1. *If α and β satisfy (1.5), then*

$$c^2 = \kappa - \mu b^2$$

for some constant κ .

The constant κ has specific geometric meaning. In order to see it, we need some discussions on wrap product.

Because β is closed, we can assume locally $\beta = df \neq 0$ for some smooth function $f(x)$. It is easy to see that the condition $b_{i|j} = ca_{ij}$ is equivalent to $\text{Hess}_\alpha f = ca^2$. According to P. Petersen's result, in this case

$$\alpha^2 = dt \otimes dt + h^2(t) \check{\alpha}^2 \quad (3.4)$$

must be locally a warped product metric on the manifold $M = \mathbb{R} \times \check{M}$, where \check{M} is an $(n-1)$ -dimensional manifold equipped with the Riemannian metric $\check{\alpha}$. Moreover, the function f depends only on the parameter t of \mathbb{R} and $h(t) = f'(t)$ [6].

Let $x^1 = t$ and $\{x^a\}_{a=2}^n$ be a local coordinate system on \check{M} , then the Riemann curvature tensor of α is determined by [1]

$$R^1_j = -\frac{h''}{h}(\alpha^2 \delta^1_j - y^1 y_j), \quad R^a_c = \check{\alpha} R^a_c - (h')^2(\check{\alpha}^2 \delta^a_c - \check{y}^a \check{y}_c) - \frac{h''}{h}(y^1)^2 \delta^a_c,$$

where $\check{y}^a = y^a$ and $\check{y}_c = \check{\alpha}_{ac} \check{y}^a$. Hence, if α is of constant sectional curvature, then combining with the first equality of (1.5) and the above two equalities we obtain $h'' + \mu h = 0$ and

$$\check{\alpha} R^a_c = [\mu h^2 + (h')^2] (\check{\alpha}^2 \delta^a_c - \check{y}^a \check{y}_c). \quad (3.5)$$

On the other hand, $\beta = df = h(t) dt$ by assumption and hence $b^2 = h^2$. Direct computations show that $b_{i|j} = h'(t) a_{ij}$, so by Lemma 3.1 we have

$$\mu h^2 + (h')^2 = \mu b^2 + c^2 = \kappa.$$

Lemma 3.2. *The Riemannian metric $\check{\alpha}$ in (3.4) is of constant sectional curvature κ .*

Next, we will show that the case when $\kappa \neq 0$ and $\mu \neq 0$ could be reduce to the case $\mu = 0$. We need some special metrical deformations for α and β , one can see [15] for details about these deformations.

Lemma 3.3. *When $\kappa \neq 0$, $\mu \neq 0$ and $\kappa - \mu b^2 > 0$, define $\bar{\alpha}$ and $\bar{\beta}$ by*

$$\bar{\alpha}^2 = \frac{|\mu|}{\kappa - \mu b^2} \left(\alpha^2 + \frac{\mu}{\kappa - \mu b^2} \beta^2 \right), \quad \bar{\beta} = \frac{|\mu|^{3/2}}{(\kappa - \mu b^2)^{\frac{3}{2}}} \beta,$$

then

$$\bar{\alpha} R^i_j = 0, \quad \bar{b}_{i|j} = \pm \sqrt{|\mu|} \bar{a}_{ij}.$$

In this case,

$$(\kappa - \mu b^2)(\kappa^{-1} + \mu^{-1} \bar{b}^2) = 1,$$

and the reversed deformations are given by

$$\alpha^2 = \frac{|\mu|^{-1}}{\kappa^{-1} + \mu^{-1} \bar{b}^2} \left(\bar{\alpha}^2 - \frac{\mu^{-1}}{\kappa^{-1} + \mu^{-1} \bar{b}^2} \bar{\beta}^2 \right), \quad \beta = \frac{|\mu|^{-3/2}}{(\kappa^{-1} + \mu^{-1} \bar{b}^2)^{\frac{3}{2}}} \bar{\beta}.$$

Proof. It is easy to see that

$$\bar{\alpha} G^i = \alpha G^i + \frac{c(x)\mu}{\kappa - \mu b^2} \beta y^i.$$

Let $\bar{Q}^i = \frac{c(x)\mu}{\kappa - \mu b^2} \beta y^i$, then

$$\begin{aligned} \bar{Q}^i_{|j} &= \mu \left(y^i y_j + \frac{\mu}{\kappa - \mu b^2} \beta y^i b_j \right), \\ y^k \bar{Q}^i_{|k.j} &= \mu \left(\alpha^2 \delta^i_j + y^i y_j + \frac{\mu}{\kappa - \mu b^2} \beta^2 \delta^i_j + \frac{\mu}{\kappa - \mu b^2} \beta y^i b_j \right), \\ \bar{Q}^i_{.k} \bar{Q}^k_{.j} &= \frac{\mu^2}{\kappa - \mu b^2} (\beta^2 \delta^i_j + 3\beta y^i b_j), \\ \bar{Q}^k \bar{Q}^i_{.k.j} &= \frac{\mu^2}{\kappa - \mu b^2} (\beta^2 \delta^i_j + \beta y^i b_j), \end{aligned}$$

where $y_i = a_{ij}y^j$. So by (2.3) we have

$$\bar{\alpha}R^i_j = \alpha R^i_j - \mu(\alpha^2\delta^i_j - y^iy_j) = 0.$$

On the other hand, direct computations show that

$$\bar{b}_{i|j} = \frac{c|\mu|^{3/2}}{(\kappa - \mu b^2)^{\frac{3}{2}}} \left(a_{ij} + \frac{\mu}{\kappa - \mu b^2} b_i b_j \right) = \pm \sqrt{|\mu|} \bar{a}_{ij}.$$

□

Notice that when $\kappa < 0$, $\bar{\alpha}^2$ is a pseudo-Riemannian metric of signature $(n-1, 1)$, because it is positive definite on the hyperplane $\beta = 0$ and negative when $y^i = b^i$. In particular, the norm of $\bar{\beta}$ with respect to $\bar{\alpha}$ is negative, i.e., $\bar{b}^2 < 0$.

$\kappa = 0$ (in this case μ must be negative by Lemma 3.1) is a very special case, because the metrical deformation given below is irreversible.

Lemma 3.4. *When $\mu < 0$ and $\kappa = 0$, define $\bar{\alpha}$ and $\bar{\beta}$ by*

$$\bar{\alpha} := \frac{\alpha}{b}, \quad \bar{\beta} := \frac{\beta}{b^2},$$

then

$$\bar{\alpha}R^i_j = 0, \quad \bar{b}_{i|j} = 0.$$

In this case, $\bar{b} = 1$.

Proof. It is easy to see that

$$\bar{\alpha}G^i = \alpha G^i - \frac{c}{b^2}\beta y^i + \frac{c}{2b^2}\alpha^2 b^i,$$

where $c = c(x)$ is a scalar function with $c^2 = -\mu b^2$. Let $\bar{Q}^i = -\frac{c}{b^2}\beta y^i + \frac{c}{2b^2}\alpha^2 b^i$. Then

$$\begin{aligned} \bar{Q}^i_{|j} &= -\frac{\mu}{2} \left(\alpha^2 \delta^i_j - 2y^iy_j - \frac{\alpha^2}{b^2} b^i b_j + \frac{2}{b^2} \beta b_j y^i \right), \\ y^k \bar{Q}^i_{|k.j} &= \mu \left\{ \left(\alpha^2 - \frac{\beta^2}{b^2} \right) \delta^i_j - \frac{\beta}{b^2} (b^i y_j - b_j y^i) \right\}, \\ \bar{Q}^i_{.k} \bar{Q}^k_{.j} &= -\frac{\mu}{b^2} (\beta^2 \delta^i_j + 3\beta b_j y^i - \beta b^i y_j - \alpha^2 b^i b_j - b^2 y^i y_j), \\ \bar{Q}^k \bar{Q}^i_{.k.j} &= -\frac{\mu}{b^2} \left\{ \left(\beta^2 - \frac{\alpha^2 b^2}{2} \right) \delta^i_j + \beta (b_j y^i - y_j b^i) - \frac{\alpha^2}{2} (b^i b_j - b_j b^i) \right\}, \end{aligned}$$

where $y_i = a_{ij}y^j$. So by (2.3) we have

$$\bar{\alpha}R^i_j = \alpha R^i_j - \mu(\alpha^2\delta^i_j - y^iy_j) = 0.$$

On the other hand, direct computations show that $\bar{b}_{i|j} = 0$.

□

4 Proof of Theorem 1.1

Proof of Theorem 1.1. Because α is of constant sectional curvature, α must be locally projectively flat due to Beltrami's theorem. Hence, there is a local coordinate system such that $\alpha G^i = \theta y^i$. By (2.2), the spray coefficients G^i of F is given by $G^i = (\theta + c\alpha\psi)y^i$.

It is easy to see that

$$\alpha_{x^k} y^k = 2\alpha\theta, \quad \beta_{x^k} y^k = c\alpha^2 + 2\beta\theta, \quad c_{x^k} y^k = -\mu\beta,$$

where the third equality is based on Lemma 3.1. Then

$$(c\alpha\psi)_{x^k} y^k = \alpha^2 [-\mu s\psi + c^2(\psi_2 + 2s\psi_1)] + 2c\alpha\theta\psi.$$

So by (2.1) we have

$$\begin{aligned}
K &= \frac{(\theta + c\alpha\psi)^2 - \theta_{x^k}y^k - (c\alpha\psi)_{x^k}y^k}{F^2} \\
&= \frac{\{\theta^2 - \theta_{x^k}y^k\} + \alpha^2\{\mu s\psi + c^2[\psi^2 - (\psi_2 + 2s\psi_1)]\}}{F^2} \\
&= \frac{\mu + \{\mu s\psi + c^2[\psi^2 - (\psi_2 + 2s\psi_1)]\}}{\phi^2}.
\end{aligned}$$

Here we use the fact that α is projectively flat and hence $\theta^2 - \theta_{x^k}y^k = \mu\alpha^2$ by (2.1). \square

In the rest of this paper, we will determine all the general (α, β) -metrics with constant flag curvature under our assumption. There are four different cases below,

- (a) $\kappa = 0$ and $\mu = 0$;
- (b) $\kappa \neq 0$ and $\mu = 0$;
- (c) $\kappa \neq 0$ and $\mu \neq 0$;
- (d) $\kappa = 0$ and $\mu \neq 0$.

As we have pointed out in Section 1, the case (a) is trivial and will not be discussed. The case (b) will be discussed in Section 6 and Section 8.

The case (c) can be reduced to the case (b) and hence it is not necessary to be discussed specially. The reason is below. If $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$ is a general (α, β) -metric satisfying Theorem 1.1 with $\kappa \neq 0$ and $\mu \neq 0$, then after deformations in Lemma 3.3, the new data $(\bar{\alpha}, \bar{\beta})$ satisfies the condition (1.8) with $\kappa \neq 0$ and $\mu = 0$. As a result, F can be reexpressed as a new form $F = \bar{\alpha}\bar{\phi}(\bar{b}^2, \frac{\bar{\beta}}{\bar{\alpha}})$. In [12], we have proved that if ϕ satisfies Equation (1.7) and (1.9), then $\bar{\phi}$ also satisfies Equation (1.7) and (1.9) with $\bar{\kappa} = |\mu|$ and $\bar{\mu} = 0$. That is to say, all the solutions provided by (c) are included naturally by in (b).

On the other hand, (d) is intrinsically different from (b). Although a data (α, β) with $\kappa \neq 0$ and $\mu = 0$ can turn to be a new data $(\bar{\alpha}, \bar{\beta})$ with $\bar{\mu} = 0$ after deformations in Lemma 3.4, (d) can not be reduced to (b) like (c). The key point is that the deformations in Lemma 3.4 is irreversible. The case (d) will be discussed in Section 7 and Section 9.

5 Solutions of Equations (1.7) and (1.9) in general case

Lemma 5.1. *The solutions of Equation (1.7) are given by*

$$\phi(b^2, s) = f(b^2 - s^2) + 2s \int_0^s f'(b^2 - \sigma^2) d\sigma + g(b^2)s,$$

where f and g are two arbitrary smooth functions.

Proof. Make a change of variables as

$$u = b^2 - s^2, \quad v = s, \tag{5.1}$$

then $b^2 = u + v^2$, $s = v$. Because

$$\frac{\partial}{\partial v}(\phi - s\phi_2) = (\phi - s\phi_2)'_1 \cdot (2s) + (\phi - s\phi_2)'_2 = 2s(\phi_1 - s\phi_{12}) - s\phi_{22} = 0,$$

there exists a smooth function $f(u)$ such that $\phi - s\phi_2 = f(b^2 - s^2)$. Let $\phi = s\varphi$, then we have $-s^2\varphi_2 = f(b^2 - s^2)$. Thus

$$\varphi = \frac{1}{s}f(b^2 - s^2) + 2 \int_0^s f'(b^2 - \sigma^2) d\sigma + g(b^2),$$

where g is a smooth function. \square

In our problem, the function $\phi(b^2, s)$ is always positive. Using the change of variables (5.1), Equation (1.9) can be reexpressed simpler as follows,

$$[\kappa - \mu(u + v^2)] \left(\frac{1}{\sqrt{\phi}} \right)_{vv} - \mu v \left(\frac{1}{\sqrt{\phi}} \right)_v + \mu \left(\frac{1}{\sqrt{\phi}} \right) - K \left(\frac{1}{\sqrt{\phi}} \right)^{-3} = 0. \quad (5.2)$$

According to the Equation 24 of Section 2.9.2 in [7], if we set $\xi = \int \frac{dv}{\sqrt{\kappa - \mu(u + v^2)}}$, then Equation (5.2) becomes

$$\left(\frac{1}{\sqrt{\phi}} \right)_{\xi\xi} + \mu \left(\frac{1}{\sqrt{\phi}} \right) - K \left(\frac{1}{\sqrt{\phi}} \right)^{-3} = 0.$$

Hence, one can obtain all the positive solutions of Equation (1.9) by solving the above equation directly.

6 Solutions of Equations (1.7) and (1.9) when $\kappa \neq 0$ and $\mu = 0$

If $\kappa \neq 0$ and $\mu = 0$, Equation (5.2) becomes

$$\left(\frac{1}{\sqrt{\phi}} \right)_{vv} = \sigma \left(\frac{1}{\sqrt{\phi}} \right)^{-3}, \quad (6.1)$$

where $\sigma := \frac{K}{\kappa}$. This equation had been solved in [12].

Lemma 6.1. [12] *The non-constant solutions of Equation (6.1) are given by*

$$\phi(u, v) = \frac{1}{p(u) \pm 2\sqrt{-\sigma}v}$$

or

$$\phi(u, v) = \frac{q(u)}{(p(u) + q(u)v)^2 + \sigma},$$

where $p(u)$ and $q(u)$ are two arbitrary smooth functions.

Lemma 6.2. [12] *When $\mu = 0$ and $\kappa \neq 0$, the non-constant solutions of Equations (1.7) and (1.9) are given by*

$$\phi(b^2, s) = \frac{1}{2\sqrt{-\sigma}} \cdot \frac{1}{\pm \sqrt{C - b^2 + s^2} \pm s}$$

or

$$\phi(b^2, s) = \frac{q(u)}{q^2(u)(Dq(u) + v)^2 + \sigma},$$

where $\sigma = \frac{K}{\kappa}$, $u := b^2 - s^2$ and $v = s$, the function $q(u) \neq 0$ is determined by the following equation

$$D^2q^4 + (u - C)q^2 - \sigma = 0,$$

where C and D are both constant numbers.

7 Solutions of Equations (1.7) and (1.9) when $\kappa = 0$ and $\mu \neq 0$

If $\kappa = 0$ and $\mu \neq 0$, Equation (5.2) is reduced to the following form

$$(u + v^2)f_{vv} + vf_v - f - \tau f^{-3} = 0, \quad (7.1)$$

where $f := \frac{1}{\sqrt{\phi}}$ and $\tau := -\frac{K}{\mu}$.

Lemma 7.1. *The non-constant solutions of Equation (7.1) are given by*

$$\phi(u, v) = \frac{2q(u)(\sqrt{u+v^2} \pm v)^2}{[q(u)(\sqrt{u+v^2} \pm v)^2 + p(u)]^2 + \tau}. \quad (7.2)$$

where $p(u)$ and $q(u)$ are two arbitrary functions.

Proof. Regard Equation (7.1) as an ODE of v . If $f_v = 0$, then f must be a constant. If $f_v \neq 0$, then multiplying the both sides of (7.1) by f_v and integrating with respect to v yields

$$(u+v^2)(f_v)^2 = 2 \int (f + \tau f^{-3}) df = f^2 - \tau f^{-2} - 2p(u),$$

where $p(u)$ is an arbitrary function of u . Since $u+v^2 > 0$ in our problem, by the above equality we have

$$\frac{df}{\sqrt{f^2 - \tau f^{-2} - 2p(u)}} = \pm \frac{dv}{\sqrt{u+v^2}}.$$

So

$$f^2 + \sqrt{f^4 - 2p(u)f^2 - \tau} = q(u)(\sqrt{u+v^2} \pm v)^2 + p(u),$$

where $q(u)$ is an arbitrary function of u . Hence, ϕ is given by (7.2). \square

Lemma 7.2. *When $\kappa = 0$ and $\mu \neq 0$, the non-constant solutions of Equation (1.7) and (1.9) are given by (7.2), where $p(u)$ and $q(u)$ satisfy an ODE system as follows:*

$$uq^2p' + (p^2 + \tau)q' = 0, \quad (7.3)$$

$$qp' - 2pq' - uqq' - 2q^2 = 0. \quad (7.4)$$

Proof. Using the change of variables (5.1), Equation (1.7) becomes

$$\phi_{vv} - 2v\phi_{uv} - 4\phi_u = 0.$$

When $\phi(u, v) = \frac{2q(u)(\sqrt{u+v^2}-v)^2}{[q(u)(\sqrt{u+v^2}-v)^2+p(u)]^2+\tau}$, with the help of Maple we know that the above equation is equivalent to the following equation

$$A_6(u)v^6 + A_4(u)v^4 + A_2(u)v^2 + A_0(u) + \sqrt{u+v^2} \{A_5(u)v^5 + A_3(u)v^3 + A_1(u)v\} = 0, \quad (7.5)$$

where

$$\begin{aligned} A_6(u) &= -A_5(u) = -32q^3(qp' - 2pq' - uqq' - 2q^2), \\ A_4(u) &= \frac{3}{2}uA_6(u) + M, \\ A_3(u) &= uA_5 - M, \\ A_2(u) &= \frac{3}{16q^2}(3u^2q^2 - p^2 - \tau)A_6 + \frac{1}{2q}(2uq + p)M, \\ A_1(u) &= \frac{3}{16q^2}(u^2q^2 - p^2 - \tau)A_5 - \frac{1}{2q}(uq + p)M, \\ A_0(u) &= \frac{1}{32q^3}[u^3q^3 - (3uq + 2p)(p^2 + \tau)]A_6 + \frac{1}{24q^2}(3u^2q^2 + 6upq + 3p^2 - \tau)M, \end{aligned}$$

and $M := 24q^2[ug^2p' + (p^2 + \tau)q']$.

Since $\sqrt{u+v^2}$ is irrational with respect to v and the remaining parts of Equation (7.5) are rational, Equation (7.5) holds if and only if

$$A_6(u)v^6 + A_4(u)v^4 + A_2(u)v^2 + A_0(u) = 0, \quad A_5(u)v^5 + A_3(u)v^3 + A_1(u)v = 0.$$

As a result, $A_i(u) = 0$ for $1 \leq i \leq 6$, which are equivalent to Equations (7.3) and (7.4).

We can obtain the same equations similarly when $\phi(u, v) = \frac{2q(u)(\sqrt{u+v^2}+v)^2}{[q(u)(\sqrt{u+v^2}+v)^2+p(u)]^2+\tau}$. \square

Lemma 7.3. *The solutions of Equations (7.3) and (7.4) with $q(u) \neq 0$ are given by*

$$p(u) = \pm\sqrt{-\tau}, \quad q(u) = \pm \frac{(C \pm \sqrt{C^2 + 8pu})^2}{4u^2} \quad (7.6)$$

or

$$p(u) = \pm \sqrt{\frac{-(C^2 - D)\tau - C(C\tau - 2u) \pm \sqrt{D(C\tau - 2u)^2 - D(C^2 - D)\tau^2}}{2(C^2 - D)}}, \quad (7.7)$$

$$q(u) = \frac{p^2 + \tau - upp' \pm \sqrt{(p^2 + \tau - upp')^2 - (p^2 + \tau)u^2p'}}{u^2p'}, \quad (7.8)$$

where C and D are constants.

Proof. (7.3) \times ($uq + 2p$) + (7.4) \times ($p^2 + \tau$) yields

$$u^2p'q^2 - 2(p^2 + \tau - upp')q + (p^2 + \tau)p' = 0. \quad (7.9)$$

When $p' = 0$, then $p = \pm\sqrt{-\tau}$ by (7.3). In this case, (7.4) is equivalent to

$$\left(u\sqrt[4]{q^2}\right)' = 2p\left(\frac{1}{\sqrt[4]{q^2}}\right)',$$

so

$$u\sqrt[4]{q^2} = 2p \cdot \frac{1}{\sqrt[4]{q^2}} + C$$

for some constant C , which leads to the solutions (7.6).

When $p' \neq 0$, then by (7.9)

$$q = \frac{p^2 + \tau}{p^2 + \tau - upp' \pm \sqrt{(p^2 + \tau - upp')^2 - (p^2 + \tau)u^2p'}}.$$

Putting the above equality into (7.3) yields

$$(p^2 + \tau)^2 p'' + (p^2 + \tau)p(p')^2 + 2\tau u(p')^3 = 0.$$

Regard u as the function of p , then the above equation turns to be

$$u'' - \frac{p}{p^2 + \tau}u' - \frac{2\tau}{(p^2 + \tau)^2}u = 0,$$

and its solutions are given below,

$$u = C(p^2 + \tau) \pm \sqrt{D}p\sqrt{p^2 + \tau},$$

where C and D are constants, and hence p can be solved and given by (7.7). Notice that the constant D here can be negative. \square

8 Proof of Theorem 1.2 and some regular examples

Proof of Theorem 1.2. It is true by Theorem 1.1 and Lemma 6.2. \square

Example 8.1-8.4 show four typical kinds of regular general (α, β) -metrics in our problem, and Example 8.5 shows that we can also give the analytic expressions in the case $\kappa \neq 0$ and $\mu \neq 0$.

Example 8.1. Take $\mu = 0$, $\lambda = 1$ in (3.1) and $\sigma = -\frac{1}{4}$, $C = 1$ in (1.10), then

$$\phi(b^2, s) = \frac{\sqrt{1 - b^2 + s^2}}{1 - b^2} \pm \frac{s}{1 - b^2},$$

and the corresponding general (α, β) -metrics

$$F = \frac{\sqrt{(1 - |x|^2 - 2\langle a, x \rangle - |a|^2)|y|^2 + (\langle x, y \rangle + \langle a, x \rangle)^2}}{1 - |x|^2 - 2\langle a, x \rangle - |a|^2} \pm \frac{\langle x, y \rangle + \langle a, x \rangle}{1 - |x|^2 - 2\langle a, x \rangle - |a|^2}$$

are locally projectively flat with constant flag curvature $K = -\frac{1}{4}$. Actually, they are just the generalized Funk metrics (1.1) expressed in some other local coordinate system.

Example 8.2. Take $\mu = 0$, $\lambda = 1$ in (3.1) and $\sigma = 0$, $C = D = 1$ in (1.11), then parts of the solutions of (1.11) are given by

$$\phi(b^2, s) = \frac{1}{\sqrt{1 - b^2 + s^2}(\sqrt{1 - b^2 + s^2} \pm s)^2} = \frac{(\sqrt{1 - b^2 + s^2} \mp s)^2}{(1 - b^2)^2 \sqrt{1 - b^2 + s^2}},$$

and the corresponding general (α, β) -metrics

$$F = \frac{\{\sqrt{(1 - |x|^2 - 2\langle a, x \rangle - |a|^2)|y|^2 + (\langle x, y \rangle + \langle a, y \rangle)^2} \mp (\langle x, y \rangle + \langle a, y \rangle)\}^2}{(1 - |x|^2 - 2\langle a, x \rangle - |a|^2)^2 \sqrt{(1 - |x|^2 - 2\langle a, x \rangle - |a|^2)|y|^2 + (\langle x, y \rangle + \langle a, y \rangle)^2}}$$

are locally projectively flat with constant flag curvature $K = 0$. Actually, they are just the generalized Berwald's metrics (1.2) expressed in some other local coordinate system.

Example 8.3. Take $\mu = 0$, $\lambda = 1$ in (3.1) and $\sigma = 1$, $C = D = 1$ in (1.11), then one solution of (1.11) is given by

$$\phi(b^2, s) = \Re \frac{1}{\sqrt{1 + 2i + b^2 - s^2 + is}},$$

and the corresponding general (α, β) -metrics

$$F = \Re \frac{|y|^2}{\sqrt{(1 + 2i + |x|^2 + 2\langle a, x \rangle + |a|^2)|y|^2 - (\langle x, y \rangle + \langle a, y \rangle)^2 + i(\langle x, y \rangle + \langle a, y \rangle)}}$$

are locally projectively flat with constant flag curvature $K = 1$. They are parts of Bryant's metrics [3, 16].

Example 8.4. Take $\mu = 0$, $\lambda = 1$ in (3.1) and $\sigma = -1$, $C = \frac{1}{2}(1 + \frac{1}{\varepsilon^2})$, $D = \frac{1}{4}(1 - \frac{1}{\varepsilon^2})$ where $0 < |\varepsilon| < 1$ in (1.11), then part of the solutions of (1.11) is given by

$$\phi(b^2, s) = \frac{1}{2} \left\{ \frac{\sqrt{1 - b^2 + s^2} + s}{1 - b^2} - \frac{\varepsilon \sqrt{1 - \varepsilon^2 b^2 + \varepsilon^2 s^2} + \varepsilon^2 s}{1 - \varepsilon b^2} \right\},$$

and the corresponding general (α, β) -metrics

$$F = \frac{1}{2} \left\{ \frac{\sqrt{(1 - |x|^2 - 2\langle a, x \rangle - |a|^2)|y|^2 + (\langle x, y \rangle + \langle a, y \rangle)^2} + \langle x, y \rangle + \langle a, y \rangle}{1 - |x|^2 - 2\langle a, x \rangle - |a|^2} - \frac{\varepsilon \sqrt{[1 - \varepsilon^2(|x|^2 + 2\langle a, x \rangle + |a|^2)]|y|^2 + \varepsilon^2(\langle x, y \rangle + \langle a, y \rangle)^2} + \varepsilon^2(\langle x, y \rangle + \langle a, y \rangle)}{1 - \varepsilon^2(|x|^2 + 2\langle a, x \rangle + |a|^2)} \right\}$$

are locally projectively flat with constant flag curvature $K = -1$. They include Shen's metrics of [8] as (39) in it.

Example 8.5. Let α and β be data satisfying (1.8) with $\mu \neq 0$ and $\kappa \neq 0$. According to Lemma 7.1 in [12], the following function

$$\phi(b^2, s) := \frac{\sqrt{|\mu|} \sqrt{\kappa - \mu b^2 + \mu s^2}}{\kappa - \mu b^2} \bar{\phi} \left(\frac{\mu b^2}{\kappa - \mu b^2} \frac{\mu}{\kappa}, \frac{|\mu|s}{\sqrt{\kappa - \mu b^2} \sqrt{\kappa - \mu b^2 + \mu s^2}} \right)$$

satisfies (1.7) and (1.9) if and only if $\bar{\phi}(b^2, s)$ is one of the functions given in (1.10) or (1.11). Hence, by Theorem 1.1 we know that the corresponding general (α, β) -metric $F = \alpha \phi(b^2, \frac{\beta}{\alpha})$ is locally projectively flat with constant flag curvature K . By the arguments in Section 3, these metrics are just the metrics in Theorem 1.2 given in a different form.

9 Proof of Theorem 1.3 and some non-regular examples

Proof of Theorem 1.3. It is true by Theorem 1.1, Lemma 7.2 and Lemma 7.3. \square

Let α and β are given by (3.1). According to (3.3), α and β satisfy (1.8) with $\kappa = 0$ if and only if

$$\lambda^2 + \mu|a|^2 = 0. \quad (9.1)$$

In this case, the length of β is given by

$$b = \frac{|\lambda - \mu\langle a, x \rangle|}{\sqrt{-\mu} \cdot \sqrt{1 - \mu|x|^2}}.$$

As a application of Theorem 1.3, some typical general (α, β) -metrics with constant flag curvature are analytic constructed below. Note that all of them are of some singularity.

$$\mathbf{K} = 0$$

Example 9.1. Let α and β be data in (3.1) with an additional condition (9.1) and take $\tau = 0$, $p(u) = 0$, $q(u) = \frac{2}{u^2}$ in (1.13), then one solution of (1.12) is given by

$$\phi(b^2, s) = (b + s)^2,$$

and the corresponding general (α, β) -metrics

$$F = \frac{\left\{ |\lambda - \mu\langle a, x \rangle| \sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y \rangle^2} + \sqrt{-\mu} (\lambda\langle x, y \rangle + (1 + \mu|x|^2)\langle a, y \rangle - \mu\langle a, x \rangle\langle x, y \rangle) \right\}^2}{-\mu(1 + \mu|x|^2)^2 \sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y \rangle^2}}$$

are locally projectively flat with vanishing flag curvature.

Example 9.2. Let α and β be data in (3.1) with an additional condition (9.1) and take $\tau = 0$, $p(u) = \frac{\sqrt{u}}{2}$, $q(u) = \frac{1}{2\sqrt{u}}$ in (1.14) and (1.15), then parts of the solutions of (1.12) are given by

$$\phi(b^2, s) = \frac{\sqrt{b^2 - s^2}}{b^2}$$

and the corresponding general (α, β) -metrics

$$F = \frac{\sqrt{b^2\alpha^2 - \beta^2}}{b^2}$$

are locally projectively flat with vanishing flag curvature. Actually, F is a positive semi-definite Riemannian metric of signature $(n - 1, 0)$.

Example 9.3. Let α and β be data in (3.1) with an additional condition (9.1) and take $\tau = 0$, $p(u) = c_1 \frac{\sqrt{1+c_1u}}{2}$, $q(u) = \frac{\sqrt{1+c_1u}(1+c_2\sqrt{1+c_1u})^2}{2u^2}$ where $c_1 = \pm 1$, $c_2 = \pm 1$ in (1.14) and (1.15), then parts of the solutions of (1.12) are given by

$$\phi(b^2, s) = \frac{\left\{ 1 + c_2\sqrt{1 + c_1(b^2 - s^2)} \right\}^2 (b + s)^2}{\sqrt{1 + c_1(b^2 - s^2)} \left\{ 1 + c_1b(b + s) + c_2\sqrt{1 + c_1(b^2 - s^2)} \right\}},$$

and the corresponding general (α, β) -metrics are locally projectively flat with vanishing flag curvature.

$$\mathbf{K} = -1$$

Example 9.4. Let α and β be data in (3.1) with an additional condition (9.1) and take $\tau = -1$, $p(u) = c_1$, $q(u) = \frac{2(1+c_2\sqrt{1+c_1u})^2}{u^2}$ where $c_1 = \pm 1$, $c_2 = \pm 1$ in (1.13), then parts of the solutions of (1.12) are given by

$$\phi(b^2, s) = \frac{(b+s)^2}{2 \left\{ 1 + c_1 b(b+s) + c_2 \sqrt{1+c_1(b^2-s^2)} \right\}},$$

and the corresponding general (α, β) -metrics are locally projectively flat with flag curvature $K = -1$.

Example 9.5. Let α and β be data in (3.1) with an additional condition (9.1) and take $\tau = -1$, $p(u) = \sqrt{1+c_1u}$, $q(u) = c_1 \frac{1}{\sqrt{1+c_1u+c_2}}$ where $c_1 = \pm 1$, $c_2 = \pm 1$ in (1.14) and (1.15), then parts of the solutions of (1.12) are given by

$$\phi(b^2, s) = \frac{b^2 - s^2}{2b \left\{ b\sqrt{1+c_1(b^2-s^2)} + c_2s \right\}},$$

and the corresponding general (α, β) -metrics are locally projectively flat with flag curvature $K = -1$.

Example 9.6. Let α and β be data in (3.1) with an additional condition (9.1) and take $\tau = -1$, $p(u) = \frac{1}{\sqrt{2}}\sqrt{1+c\sqrt{1-u^2}}$, $q(u) = -\frac{\sqrt{2}\sqrt{1+c\sqrt{1-u^2}}+u}{\sqrt{2}(1+c\sqrt{1-u^2})^{\frac{3}{2}}}$ where $c = \pm 1$ in (1.14) and (1.15), then parts of the solutions of (1.12) are given by

$$\phi(b^2, s) = \frac{2\sqrt{2} \left(\sqrt{2}\sqrt{1+c\sqrt{1-(b^2-s^2)^2}} + b^2 - s^2 \right) \left(1 + c\sqrt{1-(b^2-s^2)^2} \right)^{\frac{3}{2}} (b+s)^2}{2 \left(1 + c\sqrt{1-(b^2-s^2)^2} \right)^3 - \left\{ \left(\sqrt{2}\sqrt{1+c\sqrt{1-(b^2-s^2)^2}} + b^2 - s^2 \right) (b+s)^2 - \left(1 + c\sqrt{1-(b^2-s^2)^2} \right)^2 \right\}^2},$$

and the corresponding general (α, β) -metrics are locally projectively flat with flag curvature $K = -1$.

$$\mathbf{K} = 1$$

Example 9.7. Let α and β be data in (3.1) with an additional condition (9.1) and take $\tau = 1$, $p(u) = \frac{1}{\sqrt{2}}\sqrt{\sqrt{1+u^2}-1}$, $q(u) = \frac{2\sqrt{\sqrt{1+u^2}-1}+\sqrt{2}u}{2(\sqrt{1+u^2}-1)^{\frac{3}{2}}}$ in (1.14) and (1.15), then one solution of (1.12) is given by

$$\phi(b^2, s) = \frac{2\sqrt{2} \left(\sqrt{2}\sqrt{\sqrt{1+(b^2-s^2)^2}-1} + b^2 - s^2 \right) \left(\sqrt{1+(b^2-s^2)^2}-1 \right)^{\frac{3}{2}} (b+s)^2}{2 \left(\sqrt{1+(b^2-s^2)^2}-1 \right)^3 + \left\{ \left(\sqrt{2}\sqrt{\sqrt{1+(b^2-s^2)^2}-1} + b^2 - s^2 \right) (b+s)^2 + \left(\sqrt{1+(b^2-s^2)^2}-1 \right)^2 \right\}^2},$$

and the corresponding general (α, β) -metrics are locally projectively flat with flag curvature $K = 1$.

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